Automates d'arbre

TD n°3 : Relations

October 10th, 2019

Exercise 1 : Closure properties - back to basics

1. Given a recognizable relation, show that all its cylindrifications and projections are recognizable. (provides explicit trees automatas)
2. Is the domain and the image of a binary relation recognizable ?
3. Given R,R' binary relations, show that $R \circ R'$ is recognizable.
4. Give an example of a n-ary relation such that its ith projection followed by its ith cylindrification does not give back the original relation.
5. On the contrary, show that ith cylindrification followed by ith projection gives back the original relation.

Solution:

1. Consider the image and inverse image of the following linear tree homomorphism :

$$h_i([f_1, \dots, f_n](t_1, \dots, t_k)) = [f_1 \dots f_{i-1} f_{i+1} \dots f_n](h(t_1), \dots, h(t_k))$$

   Else, the automatas can easily be constructed.
2. The domain and the image are projections of the relation.
3. $R \circ R' = \{(t_1, t_2) \mid \exists t', (t_1, t') \in R \wedge (t', t_2) \in R'\}$. Use the constructions for $\wedge$ and $\exists$ to construct the new relation.
4. Let $\mathcal{F} = f(2), g(2), 0$. Let $t = f(0, 0)$, and $R = \{(t, t, t, t, t)\}$. 3-rd projection followed by cylindrification on R yields $R' = \{(t, t, t', t, t) \mid t \in T(\mathcal{F})\} \neq R$. For morphisms, $\forall t, h_i(h_i^{-1}(t))) = t$, while it is not true for $h_i^{-1}(h_i(t)))$.

Exercise 2 : Some relations

1. Let $\mathcal{F} = \{0(2), 1(2), n(0)\}$. Give an automaton recognizing :
   $R_1 = \{(t, t') | t, t' \in T(\mathcal{F}), Pos(t) = Pos(t') \ \wedge \forall p \in Pos(t), t(p) = 1 \Rightarrow t'(p) = 1\}$
2. Let $\mathcal{F} = \{f(2), g(1), a(0)\}$. Is the relation $R_2 = \{(g(t), t) \mid t \in T(\mathcal{F})\}$ recognizable ? And if $\mathcal{F} = \{g(1), a(0)\}$?
3. Here assume that $\mathcal{F} = \{g(1), a(0)\}$. Is R_2^* recognizable ?
4. Is $R_3 = \{(t, f(t, t')) \mid t, t' \in T(\mathcal{F})\}$ recognizable ?
5. Design two relations, one recognizable and one which is not. Challenge your classmates with them.

Solution:

1. It is the same automaton as the one for set inclusion.
2. No by the pumping lemma. Yes, similar to words.
3. Yes.
4. No, pumping lemma once again.

Exercise 3 : Rewriting systems

Let $\mathcal{F} = \{a_i(1) \mid 1 \leq i \leq n\} \cup \{0(0)\}$.

1. Prove that any rewrite system $\rightarrow$ (i.e. the one step rewriting relation) on $\mathcal{F}$ is recognizable.
2. Prove that $S = \{(a_1^k(a_1(a_2(a_2^l(0)))), a_1^k(a_2^l(0))) \mid k, p \in \mathbb{N}\}$ is recognizable.
3. Prove that S^* is not recognizable.

Solution:

1. We do the case where $\rightarrow = \{(s,t)\}$.
   - If $|s| < |t|$. The following top-down automata works :
     - $Q = \{q_{init}\} \cup \{q_k \mid 0 \leq k \leq |s|\} \cup \{q_{|s|+k}^{\alpha_1,\dots,\alpha_k} \mid 0 \leq k \leq |t| - |s|, \alpha_i \in \mathcal{F} \cup \{\#\}\} \cup \{q_f^{\alpha_1,\dots,\alpha_{|t|-|s|}} \mid \alpha_i \in \mathcal{F} \cup \{\#\}\}$
     - $I = \{q_{init}\}$
     - $\Delta =$
       - $q_{init}(a_i a_i(x)) \longrightarrow q_{init}(x)$
       - $q_{init}(x) \longrightarrow q_0(x)$
       - $q_k(s_k t_k(x)) \longrightarrow q_{k+1}(x)$
       - $q_{|s|}(x) \longrightarrow q_{|s|}^{\varnothing}(x)$
       - $q_{|s|+k}^{\alpha_1,\dots,\alpha_k}(\alpha_{k+1} t_{|s|+k}(x)) \longrightarrow q_{|s|+k+1}^{\alpha_1,\dots,\alpha_{k+1}}(x)$
       - $q_{|t|}^{\alpha_1,\dots,\alpha_{|t|}}(x) \longrightarrow q_f^{\alpha_1,\dots,\alpha_{|t|}}(x)$
       - $q_f^{\alpha_1,\dots,\alpha_k}(\alpha_{k+1}\alpha_1(x)) \longrightarrow q_f^{\alpha_2,\dots,\alpha_{k+1}}(x)$ if $\alpha_1 \in \{a_i \mid 1 \leq i \leq n\}$
       - $q_f^{0,\#,\dots,\#}(\#0) \longrightarrow$
   - If $|s| \geq |t|$. Idem.
2. Use question 3.
3. No because $S^* \cap T(\mathcal{F}) \times \{0\} = \{(a_1^n(a_2^n(0)), 0) \mid n \in \mathbb{N}\}$ which is not recognizable by the pumping lemma.